Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

BAKALÁŘSKÁ PRÁCE

Hana Štěpánková

Algebraické invarianty v teorii uzlů

Katedra algebry

Vedoucí bakalářské práce: RNDr. David Stanovský, Ph.D.
Studijní program: Matematika, MMIB

2010

Na tomto místě bych ráda poděkovala především vedoucímu práce Davidovi Stanovskému za všechny jeho rady, návrhy a připomínky. Také bych ráda poděkovala Lucienu Holmesovi, který ochotně a trpělivě opravoval jazykové nedostatky této práce.

Prohlašuji, že jsem svou bakalářskou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 19. 7. 2010 Hana Štěpánková

Contents

Název práce: Algebraické invarianty v teorii uzlů
Autor: Hana Štěpánková
Katedra (ústav): Katedra algebry
Vedoucí bakalářské práce: RNDr. David Stanovský, Ph.D.
e-mail vedoucího: david.stanovsky@gmail.com

Abstrakt: V předložené práci představujeme základy teorie uzlů, oblasti algebraické topologie, která se zabývá matematickými uzly. Matematické uzly jsou podobné obyčejným uzlům, které můžeme uvázat na kousku provázku. Konce tohoto provázku ale spojíme, aby se uzel nemohl rozvázat. Základní problém v teorii uzlů je určit, kdy jsou dva uzly ekvivalentní. Jinak řečeno, je možné předělat jeden uzel ve druhý, aniž bychom přestříhli provázek, ze kterého je uzel vyroben? Funkce, která dvěma ekvivalentním uzlům přiřadí vždy stejnou hodnotu, se nazývá uzlový invariant. V této práci se zaměřujeme na algebraické invarianty – maticové invarianty a polynomy. Nejprve představíme několik základních pojmů z teorie uzlů, a poté předložíme teorii nezbytnou pro pochopení řešených cvičení, která jsou vždy na konci dané kapitoly.

Klíčová slova: matematický uzel, uzlový invariant, Seifertova matice, Alexandrův polynom

Title: Algebraic invariants in knot theory
Author: Hana Štěpánková
Department: Department of Algebra
Supervisor: RNDr. David Stanovský, Ph.D.
Supervisor's e-mail address: david.stanovsky@gmail.com

Abstract: In this thesis we study the fundamentals of knot theory: an area of algebraic topology that studies mathematical knots. A mathematical knot is very much like a real-life knot that we can tie on a piece of string, except we glue the ends together so that the knot cannot be untangled. The most basic question in knot theory is when two knots are equivalent. In other words, can we make one knot into the other without cutting the string? The functions that assign any two equivalent knots the same value are called knot invariants. The focus of this thesis is on the algebraic invariants: matrix invariants and various polynomials. At first we introduce some basic definitions from knot theory, followed by the theory necessary for understanding the solved exercises at the end of each chapter.

Keywords: mathematical knot, knot invariant, Seifert matrix, Alexander polynomial

Introduction

Knot theory is an area of geometric topology which examines mathematical knots and links. Formally speaking, a knot is an embedding of a circle in 3-dimensional Euclidean space. We can also imagine it as a piece of tangled string with both ends glued together. Link is a disjoint union of one or more knots (components).

Knots are usually represented by planar diagrams. The most basic question in knot theory is, when do two diagrams represent the same knot? How can we tell that the knot with a certain diagram cannot be untangled without cutting the string? In other words, how do we know whether a knot is in fact the trivial knot (the unknot, the circle)?

Useful tools for distinguishing knots are knot invariants. A knot invariant can be any property of a knot that remains the same no matter which diagram of the knot we choose. Knot invariants can be of different nature, from the basic geometric ones like the unknotting number or polygonal index to more sophisticated ones, e.g., matrix invariants or polynomials.

The goal of this thesis was to study the fundamentals of the knot theory in order to be able to continue the research in this field on the graduate level. To achieve this I have read two books [1, 2] and solved a large number of exercises from these books. A selection of them is the principal content of this thesis.

The thesis is divided into four chapters. First, some basic definitions from the knot theory are introduced. The following chapters cover the fundamental definitions and theorems leading to the polynomial invariants. At the end of each chapter, there is a selection of solved exercises that demonstrates the use of the terms and theorems stated in the rest of the chapter.

In order to keep the document within a reasonable length, the thesis is not entirely self-contained. I included mostly the theory necessary for understanding the exercises and their solutions; often using an informal definition rather than exact mathematical terms. I omitted most of the proofs; the ones included are designed to show the general solving process for the given problem. The formal definitions and proofs are referenced in the books. I did not include the exercises that do not relate directly to the thesis topic, are repetitive or purely graphical.

Chapter 1

Basic Facts from Knot Theory

As was mentioned above, a **knot** $K \subset \mathbb{R}^3$ is a subset of points homeomorphic to a circle. What it means is that a knot is a piece of tangled string with the two ends glued together. A **link** is just a union of knots (components); they can be tangled together or disjoint. A link can have one or more components, which means that knots are just a special group of links with one component. Each of these components can be assigned an orientation; we then talk about oriented links.

The simplest example of a knot is the trivial knot, sometimes called the unknot. It is an unknotted circle denoted by $\bigcirc$.

In a plane, links are represented by **diagrams**: projections of links where one strand passing under another is shown as if it was interrupted (these are called crossings). We say that that two links are equivalent (or more formally that two diagrams represent the same link) if we can make one link from a piece of string, connect the ends and remake it into the other one without cutting the string. This is called *ambient isotopy* (for the exact definition, see [1, p.4]).

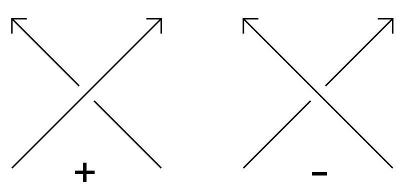

Figure 1.1: A diagram of a positive and a negative crossing in an oriented graph.

A link is called a **split link** if it has two or more components and there is a sphere embedded in $\mathbb{R}^3 \setminus L$ so that there are components of L on each side of the sphere.

By switching all the crossings in the link L we obtain its mirror image, denoted by L^*. We say that L is **amphicherial** if $L = L^*$.

It is actually a very difficult problem to decide whether two knots are distinct. We have numerous functions called **link invariants**; these are functions from the set of links that gives the same answer for any projection of the given link. To show that a function is in fact a link invariant we mostly use the following theorem:

Theorem. *Any two diagrams of a link are connected with a sequence of the Reidemeister moves.*

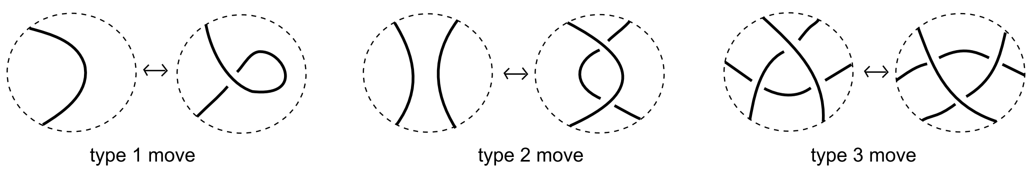

Figure 1.2: The Reidemeister moves.

If the value of the function stays the same when we modify the link by any of these moves, the function is a link invariant.

Some invariants are easy to define but very difficult to determine. An example of that can be the **crossing number** of a link. It is a minimum number of crossings in any projection of a given link. But how can we tell without checking all the projections of the link that there is no projection with fewer crossings than we alerady have? This is the problem with most invariants that minimise some geometric property of a knot.

The last terms we need to define are prime and composite links. Let L be a link and let S be a sphere which meets L in exactly two points. We can divide L into two components, one inside and one outside the sphere. If we connect the loose ends of each component, we obtain two new links. These links are called proper factors of L if they are not trivial and not equal to the link itself. A link is **prime** if it is non-trivial, non-split and has no proper factors. A link is **composite** if it has proper factors.

In some exercises we examine special classes of links called torus and pretzel links. A torus link can be embedded in $\mathbb{R}^3$ so that it lies on the surface of an unknotted torus. Depending on the parity of the parameter p, the $(p, 2)$ torus link has one or two components. A pretzel link is a link with three boxes of half-twists. All the parameters p, q and r can be positive or negative. Examples are shown in Figure 1.3.

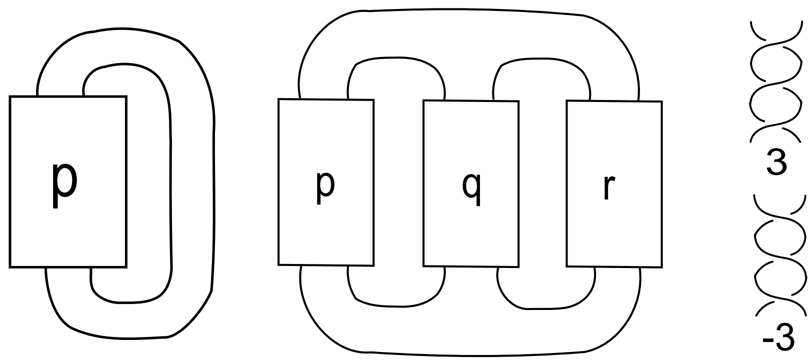

Figure 1.3: General diagrams of $(p, 2)$ torus knot and (p, q, r) pretzel knot with examples of three positive and three negative half-twists.

For the exact definitions of the terms defined in this section see chapters 1, 3 and 4 of [1].

Chapter 2

Surfaces and Knots

For any given link L, there is an orientable surface F whose boundary is the link L. We say that F *spans* L, or that F is the *spanning surface* for L.

2.1 Properties of Surfaces

First, let us introduce some basic properties of surfaces in general. While studying knots, we are interested in connected orientable surfaces with one or more boundary components.

The surface is orientable when we can color it with two different colors and these colors do not meet anywhere except at the boundary of the surface.

We can measure the connectedness of the surface F by the maximum number of non-intersecting, non-separating loops on F; that is how many different loops there are, so that when we cut along the loop the surface does not come apart. On orientable surfaces, these loops always occur in pairs ([1, p.39]). The number of these pairs is a property of the surface called the *genus* of the surface, marked as *g(F)*.

The genus together with the number of boundary components, $|\partial F|$, is sufficient to distinguish surfaces that are not homeomorphic ([1, p.39]).

Every orientable surface can be triangulated. That means that the surface can be cut into a certain number of triangles (finite or infinite). We are interested only in surfaces with finite triangulation: those are called compact spaces. An example of compact space is a sphere or a torus.

We define the Euler characteristics of a surface F with a following formula:

$$\chi(F) = V - E + T$$

where V is the number of vertices, E is the number of edges and T is the number of triangles in the triangulation of F.

Theorem 2.1.1. *The Euler characteristics of the surface is independent of the chosen triangulation and is related to the genus as follows:*

$$2g(F) = 2 - \chi(F) - |\partial F|$$

A small technical detail: the surface does not have to be divided into triangles in order to compute its Euler characteristic. The faces can be any polygons, just as long as their boundaries are made up of a sequence of edges connected with vertices.

For more detailed definitions and proofs see [2, p.74-82] or [1, p.39-41].

2.2 Seifert Surfaces

Definition 2.2.1. (link genus) The *genus* of an oriented link L is the minimum genus of any connected orientable surface whose boundary is ambient isotopic to L. The genus of an unoriented link is the minumum genus over all possible orientations. The genus of L is denoted by $g(L)$.

The first problem with this definition is obvious, as with other geometric properties: unless we list the genera of all the possible surfaces spanning L, it is rather difficult to show that the genus of the particular surface is really minimal for L.

The other problem is whether the link genus is well defined: is there an orientable surface that has L for its boundary for every L? The answer to this question lies in the following algorithm: it yields an orientable spanning surface F for any link L.

Seifert's algorithm

1. If link L is not oriented, we choose the orientation of each component.

2. We eliminate all the crossings as shown in Figure 2.1. In each crossing, there are two strands coming in and two strands going out. The crossing is eliminated by re-connecting each incoming strand to the adjacent outgoing strand, with respect to the orientation. This is also called smoothing the crossing.

3. The output of the second step is at least two oriented loops called the *Seifert circles*. All these circles bind discs in a plane. We assign different heights to the discs that appear to lie on top of each other. That way all the discs are disjoint.

4. We connect the discs with half-twisted bands in places of the crossings that we eliminated in step one. The direction of the twist is chosen according to the original crossing.

5. Now we choose the orientation (coloring) of the surface. The loops have orientation inherited from the original diagram. We paint the upward side of discs with clockwise orientation dark, the downward side light. We switch the colors for discs with counter-clockwise orientation. The twisted bands are colored with respect to the discs they are connecting.

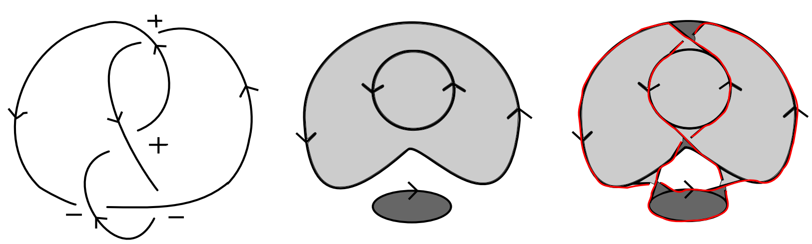

Figure 2.1: Application of the Seifert algorithm on a knot

The coloring of the twisted bands described in step 4 agrees with the coloring chosen for the discs. Any two discs that are above each other are of the same orientation, so the color on the upward face of the upper disc spreads over the twisted band to the upward face of the lower disc. The same goes for the downward face. Two discs next to each other are of the opposite orientation and the twisted band connects the upward face of one disc to the downward face of the other disc.

The output of this algorithm is an orientable surface spanning the link L. This surface is called the projection surface, or the *Seifert surface for L*. A detailed description can be found in [1, p.103] or [2, p.96].

Theorem 2.2.2. *The Euler characteristic of a Seifert surface F constructed from a diagram D is*

$$\chi(F) = s - c,$$

where s is the number of Seifert circles and c is the number of crossings in D.

Corollary 2.2.3. *The genus of a connected Seifert surface constructed from a diagram D is*

$$2g(F) = [1 - s + c] + [1 - \mu(D)],$$

where $\mu(D)$ is the number of components in D.

For the proofs see Exercise 2.2.

Theorem 2.2.4. *Let K_1, K_2 be knots, then*

$$g(K_1 \# K_2) = g(K_1) + g(K_2).$$

The proof can be found in [2, p.100]

2.3 Seifert Graphs

We can constuct a graph G from the Seifert surface for any link L. We call this graph the *Seifert graph* of L. The vertices of G are the Seifert circles and the edges of G the twisted bands connecting the corresponding circles. We can assign + or - to each edge according to the sign of the original crossing. This graph carries much of the information about the original surface within itself. However, it is not enough to reconstruct it just from the graph: the graph does not tell us anything about the mutual positions of the circles. The example is shown in Figure 2.2: the Seifert graph has been constructed from the Seifert surface shown in Figure 2.1 but can be reconstructed into a surface with three boundary components.

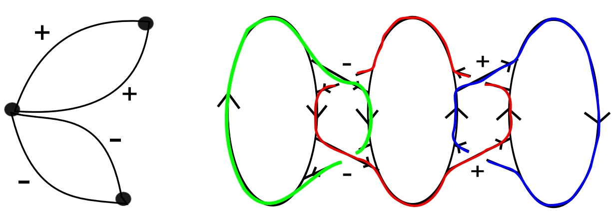

Figure 2.2: Seifert graph and one of its possible reconstructions

2.4 Exercises

Exercise 2.1. ([2, p.82] Ex. 4.9)
Show that for a surface without boundary with genus g,

$$\chi(F) = 2 - 2g.$$

Solution. The genus of F together with the number of boundary components is enough to distinguish any two surfaces that are not homeomorphic. Here we consider only surfaces with no boundary: therefore for a surface with genus g, we take g tori connected to form a chain (connected sum of the tori). The genus of a torus is one (there is one pair of non-separating loops). Each torus connected to the chain adds a pair of non-separating loops, therefore increases the genus of the surface by one.

1. g = 1: The Euler characteristic of the torus is zero [2, p.75] and therefore the equality for $g = 1$ stands.

2. Let T_n be a connected sum of n tori and g_n its genus, then we perform the following induction step:

   $$\chi(T_n) = 2 - 2g_n \Rightarrow \chi(T_{n+1}) = 2 - 2g_{n+1}$$

   Connecting one more torus to T_n increases its genus by one:

   $$g_{n+1} = g_n + 1.$$

   How does the Euler characteristic change? To connect the tori, we have to remove a triangle from each surface and connect them by the boundaries of the triangles. The resulting surface has two fewer triangles, 3 fewer edges and three fewer vertices than the sum of all these components in the two original surfaces. Because in the formula for $\chi(F)$ the edges are added and the vertices subtracted, the Euler characteristic of T_n can be computed as follows:

   $$\chi(T_{n+1}) = \chi(T_n)+\chi(T)-(3-3+2) = 2-2g_n+0-2 = 2-2(g_n+1) = 2-2g_{n+1},$$

   which is what we set out to prove.

A surface F with genus g with n boundary components is homeomorphic to a connected sum of g tori with n discs removed. Each of the removed discs can be considered as removing the interior of one triangle; therefore the Euler characteristic goes down by one for each boundary component:

$$\chi(F) = \chi(T_g) - |\partial F|$$

$\square$

Exercise 2.2. ([2, p.98] Ex. 4.19)

Let F be the Seifert surface for the knot K. Show that if c is the number of crossings and s is the number of Seifert circles in F, then:

$$\chi(F) = s - c$$

and the genus of the surface is

$$g = (c - s + 1)/2.$$

Solution. We divide F such that we put an edge across each twisted band (crossing). The number of faces is obviously the same as the number of the Seifert circles; the number of vertices is twice the number of crossings. The edges are a little complicated. We have:

- edges going across the twisted bands: we get one for each crossing, and
- edges going along the faces: a circle with n twisted bands attached is subdivided into n arcs separated by those bands. Therefore, we get two edges corresponding to a twisted band, one on each of the circles that the band connects.

All can be summarized as follows:

$$V = 2c$$

$$E = c + 2c = 3c$$
$$T = s;$$

so from the definition of the Euler characteristic we get:

$$\chi(F) = 2c - 3c + s = s - c. \tag{2.1}$$

To compute the genus of F, we will use the formula obtained in the previous exercise. The surface has one boundary component because its boundary is a knot. From Theorem 2.1.1 and (2.1) we get:

$$2g = 2 - (s - c) - 1,$$

and dividing by two gives us the desired result:

$$g = \frac{c - s + 1}{2}$$

□

Exercise 2.3. ([2, p.104] Ex. 4.29)

Prove that if we take the composition of n copies of the same non-trivial knot, calling the result J_n, the crossing number of J_n is at least $2n + 1$.

Solution. Because J is non-trivial, its genus must be at least 1: $g(J) \geq 1$. According to Theorem 2.2.4, the genus of a composite knot is the sum of its factors:

$$g(J\#J) = g(J) + g(J),$$

which means that for n factors we get:

$$g(J_n) = n \cdot g(J) \geq n \tag{2.2}$$

Using the formula from the previous exercise, we obtain the following:

$$c = 2g + s - 1.$$

We know that the Seifert algorithm applied on a non-trivial knot always generates at least two circles: $s > 2$. With that and (2.2), we get:

$$c \geq 2n + 2 - 1 = 2n + 1$$

□

Chapter 3

Matrix Invariants

In this chapter we will explore somewhat stronger and more sophisticated link invariants, link signature and link determinant. With these tools, we can prove that there are non-trivial knots. Unlike the simpler invariants based on minimizing a geometrical property, these are computed directly from any surface spanning the given link. The price we have to pay is a more complicated definition. First we need to introduce a few basic facts about homology groups.

3.1 A Few Facts from Homology Theory

Let G be a connected oriented graph with a set of vertices V and a set of edges E. Each edge is denoted as an ordered pair of vertices, $e = [v, w]$. We define $C_1(G)$ as a set of all formal linear combinations of edges in G with coefficients taken from the abelian group $(\mathbb{Z}, +)$. The elements of this set are called *1-chains*.

We define the addition of two 1-chains as follows:

$$\sum_{i=1}^{n} \lambda_i e_i + \sum_{i=1}^{n} \mu_i e_i = \sum_{i=1}^{n} (\lambda_i + \mu_i) e_i.$$

The set of 1-chains together with the addition operation makes an abelian group denoted by $C_1(G)$. We define the boundary operator $\partial(C_1)$ as a linear map:

$$\partial(\sum_{i=1}^{n} \lambda_i e_i) = \sum_{i=1}^{n} \lambda_i \partial(e_i)$$

where for $e = [v, w]$, $\partial(e) = w - v$.

Every path or circuit in G has a corresponding 1-chain. The coefficients denote how many times the path follows the particular edges; the sign of the coefficient marks the direction of the path compared to the orientation of the edge. This means that we can get any path or circuit in G not taking its orientation into consideration: if we need a particular edge with the opposite orientation, we give it a negative sign in the linear combination.

The 1-chains with no boundary are called *1-cycles*: for example, every circuit in unoriented G would be a 1-cycle. The subgroup of 1-cycles is denoted by $Z_1(G)$. We can construct the basis of $Z_1(G)$ as follows ([1, p.131]):

1. Choose T, a spanning tree for unoriented G.

2. Number the edges that are not in T as $e_1, ..., e_r$.

3. For each $e_i, 1 \leq i \leq r$, there is a circuit in $T \cup \{e_i\}$. Let z_i be the corresponding 1-cycle in $Z_1(G)$ so that the coefficient of e_i is $+1$.

This algorithm gives us a set of r 1-cycles which is a basis for $Z_1(G)$.

This construction can be generalized to sets of n-dimensional objects called *simplicial complexes*. The elements are called simpleces. Examples of such sets include triangulated surfaces or graphs.

We are interested in 2-simpleces (triangles). Accordingly to the construction of a group of 1-chains, we can get a group $C_2(X)$ of 2-chains where the elements are all formal linear combinations of oriented 2-simpleces in a simplicial complex X.

The boundary operator for 2-chains is defined as a linear map on C_2 where the boundary of a triangle is defined as the sum of its edges. Note that because the boundary of a 2-chain is a linear combination of edges, it is a 1-chain. We define $B_1(X)$ as a set of 1-chains that are boundaries of 2-chains. These 1-chains are always circuits, so $B_1(X)$ is a normal subgroup of $Z_1(X)$. The quotient group $H_1(X) = Z_1(X)/B_1(X)$ is called the first homology group of X.

For any compact surface we can find a finite triangulation. That means that we can consider it as a simplicial complex and find a homology group for it. The homology group is in fact independent of the triangulation chosen and we denote it by $H_1(F)$ ([1, p.137]).

Let us look at the basis of this group. The triangulation of F is finite, therefore H_1 is finitely generated. Any loop that binds a disc in F is trivial:

it is a boundary. Any separating loop in F when F is a closed surface is also trivial: it is the boundary of the 2-chains produced by the separation. The only non-trivial elements of $H_1(F)$ for a closed surface are the non-separating loops. For surfaces with boundary, there is one non-trivial loop for each boundary component after the first one ([1, p.137]).

Theorem 3.1.1. *The size of the basis for* $H_1(F)$ *is* $2g + \mu - 1$.

Let us consider the Seifert surface F of a link L. We can remove the triangles from the rims of the discs and twisted bands so that the surface remains connected until the discs and bands are reduced to dots and lines, respectively. This is exactly the Seifert graph G for the link L. The graph was constructed from F just by removing trivial discs. Therefore, their homology bases are isomorphic ([1, p.138]).

Applying the algorithm described above on G, we find a homology basis for F. With that we can proceed to defining the Seifert matrix for a link. For details and proofs see [1], chapters 6.1-6.4.

3.2 Seifert Matrix and Link Invariants

Definition 3.2.1. (linking number) Let L be a 2-component link, $L = K_1 \cup K_2$. Let $c_1, ..., c_n$ be the crossings of K_1 with K_2. We assign ± 1 to each c_i according to Figure 1.1. The linking number of K_1 and K_2 is defined as follows:

$$lk(K_1, K_2) = \frac{1}{2} \sum_{i=1}^{n} c_i$$

Definition 3.2.2. (Seifert matrix) Let F be the projection surface of the link L. Let $b : F \times [0, 1] \to \mathbb{R}^3$ be a homeomorphism such that $b(F \times \{0\}) = F$ and $b(F \times \{1\}) = F^+$ lies on the positive side of F. Let $\{a_1, ..., a_n\}$ be the homology basis for F and a_i^+ the loop a_i raised so it lies on F^+. We define the *Seifert matrix* for L as follows:

$$A = \left\{ lk(a_i, a_j^+) \right\}_{n,n}$$

Definition 3.2.3. (link determinant) The determinant of a link L, denoted by $\det(L)$, is the absolute value of $\det\left(M + M^T\right)$, where M is any Seifert matrix for L.

Definition 3.2.4. (link signature) The signature of a link L, denoted by $\sigma(L)$, is the signature of $M + M^T$, where M is any Seifert matrix for L.

By the signature of a matrix we mean the number of positive entries minus the number of negative entries of the matrix in the diagonal form.

The signature and determinant of a link L are link invariants independent of the diagram of L chosen for their calculation. The proof can be found in [1, p.144].

3.3 Exercises

Exercise 3.1. ([1, p.155] Ex. 6.9.6.)

Let K be a *(p, q, r)* pretzel knot with p, q and r all odd. Find a Seifert matrix for K. Compute $\det(K)$ and $\sigma(K)$.

Solution. First we determine the size of the Seifert matrix M: according to Theorem 3.1.1., the size of the basis for $H_1(F)$ is $2g + \mu - 1$. With K being a knot we get the number of components $\mu = 1$; the second equality is the result of Exercise 2.2:

$$2g + 1 - 1 = 2g = c - s + 1,$$

According to Figure 3.1, the Seifert surface of K has $2 + (p-1) + (q-1) + (r-1)$ faces and $p + q + r$ crossings, therefore

$$2g = p + q + r - (2 + (p-1) + (q-1) + (r-1)) + 1 = 2$$

We find these two loops with the algorithm described above: the loop a is the circuit we get by adding the edge e to the spanning tree, the loop b by adding f.

The linking number of a loop and its positive image is half the sum of their crossing number; the linking number of a and b depends only on q:

- $lk(a, a^+) = \frac{p+q}{2}$
- $lk(b, b^+) = \frac{q+r}{2}$
- $lk(a, b^+) = \frac{q+1}{2}$
- $lk(b, a^+) = \frac{q-1}{2}$

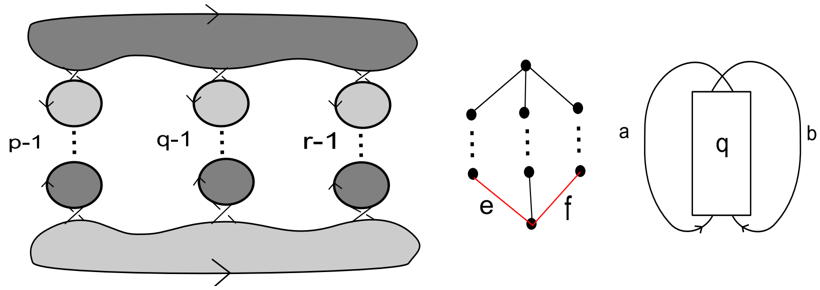

Figure 3.1: The Seifert surface for the (p, q, r) knot and its Seifert graph. The loops a and b form the basis for F.

Now we can determine the Seifert matrix for K and add it to its transpose:

$$M+M^T = \begin{pmatrix} \frac{p+q}{2} & \frac{q+1}{2} \\ \frac{q-1}{2} & \frac{q+r}{2} \end{pmatrix} + \begin{pmatrix} \frac{p+q}{2} & \frac{q-1}{2} \\ \frac{q+1}{2} & \frac{q+r}{2} \end{pmatrix} = \begin{pmatrix} p+q & q \\ q & q+r \end{pmatrix} \sim \begin{pmatrix} p & -r \\ q & q+r \end{pmatrix}$$

From that we can calculate the determinant of K:

$$\det(K) = \left|det(M + M^T)\right| = |p(q + r) + qr| = |pq + pr + qr|$$

We compute the knot signature with the subdeterminant method ([1, p.152]):

- $\Delta_0 = 1$
- $\Delta_1 = p$
- $\Delta_2 = p(q + r) + rq = pq + pr + qr,$

and so

$$\sigma(K) = \text{sign}\,(p) + \text{sign}\,(p \cdot (pq + pr + qr))$$
$$\sigma(K) = \text{sign}\,(p) \cdot (1 + \text{sign}\,(pq + pr + qr)),$$

This gives us two possible values of signature:

1. if $pq + pr + qr < 0$ then $\sigma(K) = 0$
2. if $pq + pr + qr > 0$ then $\sigma(K) = 2 \cdot \text{sign}\,(p)$

These are all the possibilities we have; $pq + pr + qr$ can never be equal to 0 because it is a sum of three odd numbers. $\square$

Exercise 3.2. ([1, p.155] Ex. 6.9.7.)

For which values of p, q and r can the determinant and signature not distinguish the *(p, q, r)* pretzel knot from the trivial knot?

Solution. We can use the results of the previous exercise. Knowing that $\det(\bigcirc) = 1$, we need p, q and r to satisfy the following equation:

$$|pq + pr + qr| = 1.$$

The signature of the trivial knot is 0, so we know that $pq + pr + qr$ must be negative:

$$pq + pr + qr \;\; = \;\; -1 \tag{3.1}$$

This equation has infinitely many different integer roots. Before we look at them we need to discuss one special case. What if one of the numbers p, q or r was chosen arbitrarily. Then we can put the remaining two equal to ± 1. These numbers would always satisfy the condition in (3.1) but unfortunately the pretzel knot with these parameters is always the unknot. Now we can search for solutions where at least two parameters are greater than one.

If we look at the congruence modulo p, we get the following result:

$$qr \equiv -1 \; (\mathrm{mod} p)$$

which we can rewrite as:

$$qr = 2ap - 1, \;\; a \in \mathbb{Z}. \tag{3.2}$$

We know that qr is odd, therefore the multiple of p in the second equation must be even. For the original equation (3.1) to be satisfied, we need the following:

$$pq + pr + qr = -1$$

we use the substitution $qr = 2ap - 1$:

$$pq + pr + 2ap - 1 = -1$$

which is equivalent to:

$$p(q + r + 2a) = 0$$

and because $p \neq 0$ we get:

$$q + r + 2a = 0$$

which can be written as:

$$r = -2a - q. \tag{3.3}$$

These facts yield a system of equations with variables q, r and parameters a, p: by the substitution $r = -2a - q$ in the equation (3.2) we get

$$q(-2a - q) = 2ap - 1$$

and after expanding and putting all terms on one side we get the desired quadratic equation in q with parameters a and p:

$$q^2 + 2aq + 2ap - 1 = 0.$$

We use the quadratic formula to solve this equation:

$$q_{1,2} = \frac{-2a \pm \sqrt{4a^2 - 4(2ap - 1)}}{2} = -a \pm \sqrt{a^2 - 2ap + 1} \tag{3.4}$$

Now we need to find out for which values of a the square root of $a^2 - 2ap + 1$ is an integer; with that we can get all solutions for (3.1):

$$a^2 - 2ap + 1 = b^2, \tag{3.5}$$

where $b \in \mathbb{Z}$. We rewrite this equation so that all the terms are on one side and apply the quadratic formula. This time a is the variable and p is the parameter.

$$a_{1,2} = \frac{2p \pm \sqrt{4p^2 - 4(1 - b^2)}}{2} = p \pm \sqrt{p^2 + b^2 - 1} \tag{3.6}$$

We are in the same situation as before: we need the square root of the discriminant to be an integer.

$$p^2 + b^2 - 1 = c^2 \tag{3.7}$$

where $c \in \mathbb{Z}$. But this time, after subtracting b^2, we can factorise the equation as follows:

$$(p - 1)(p + 1) = (c - b)(c + b), \tag{3.8}$$

from which we can with p given as odd compute c and b.

Now let us summarise the process of finding q and r with p given, where all the numbers are odd and satisfy the equation (3.2).

1. For a p given as odd, we compute the product $x = (p-1)(p+1)$.
2. We find a different factorisation of x, e.g. $x = yz$, where both y and z are even.
3. From this and (3.8) we get a system of two linear equations that are easy to solve:

$$c - b = y$$
$$c + b = x.$$

   With the values of b and c we can proceed to computing q and r.

4. Putting the substitution in (3.7) into (3.6) gives us two values of a:

$$a_{1,2} = p \pm c.$$

5. We use the substitution (3.5) in the equation (3.4) and for each value of a we get two values of q:

$$q_{1,2} = -a \pm b = -p \mp c \pm b.$$

   For each of these values we can compute the corresponding r from the relation given by (3.3).

This means that for each value of p and each factorisation described in point two, we get four ordered pairs of values for (q, r) and these are all the possible solutions for (3.1).

□

Exercise 3.3. ([1, p.155] Ex. 6.9.8.)

Show that the signature of the $(p,\ 2)$ torus knot with $p \geq 3$ is p - 1.

Solution. From Figure 3.2 we can see that there are only two Seifert circles for any number of crossings p:

- s = 2
- c = p

According to Theorem 3.1.1, the number of loops in the homology basis is $2g + \mu + 1$. The first equality comes from Theorem 2.1.1 and the fact that K is a knot; the second from Theorem 2.2.3:

$$2g + \mu - 1 = 1 - \chi = 1 - (2 - p) = p - 1$$

The basis chosen accordingly to the spanning tree of the Seifert graph is not sufficient (the spanning tree here consists of only one edge, e). Let us denote the loop obtained by adding the edge e_i to the spanning tree as l_i. We choose the basis $\{a_i\}_{i=1}^{p-1}$ such that $a_1 = l_1$ and $a_i = l_i - l_{i-1}$ for $i \in 2, \ldots p-1$. These loops make up a chain (as shown in Figure 3.2); all the loops are oriented clockwise.

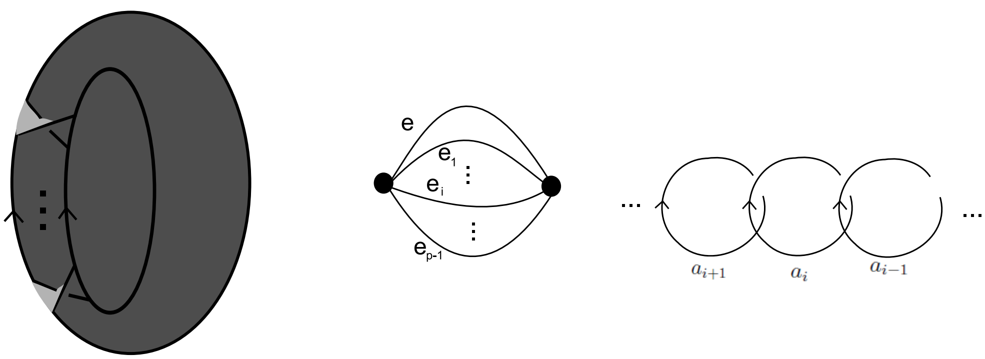

Figure 3.2: Seifert surface for a $(p, 2)$ torus knot, its Seifert graph and a part of the new basis.

Because p is positive, the linking numbers are the following:

- $lk(a_i, a_{i+1}^+) = -1$
- $lk(a_{i+1}, a_i^+) = 0$
- $lk(a_i, a_i^+) = 1$
- $lk(a_i, a_j^+) = 0$ for $|i - j| > 1$

$$M = \left(\begin{array}{c|cccccc} & a_1^+ & a_2^+ & a_3^+ & \cdots & a_{p-1}^+ & \\ \hline a_1 & 1 & -1 & 0 & \cdots & \cdots & 0 \\ a_2 & 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_{p-3} & 0 & \cdots & 0 & 1 & -1 & 0 \\ a_{p-2} & 0 & \cdots & \cdots & 0 & 1 & -1 \\ a_{p-1} & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{array}\right)$$

$$M + M^T = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Let us denote the matrix A_n where $n = p - 1$ is its size.

We compute the determinant of A_n first using the Laplace expansion along the first column, then again along the first row:

$$\det A_n = 2 \cdot \det A_{n-1} + \left|\begin{array}{c|ccc} -1 & 0 & \cdots & 0 \\ \hline -1 & & & \\ 0 & & A_{n-2} & \\ \vdots & & & \\ 0 & & & \end{array}\right| = 2 \cdot \det A_n - \det A_{n-2}$$

This leads to the following recurrence equation:

$$\det A_n - 2 \cdot \det A_{n-1} + \det A_{n-2} = 0$$

which has the characteristic polynomial:

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

with one double root $\lambda = 1$.

Therefore the solution of the recurrence equation is in the following form:

$$\det A_n = a \cdot 1^n + b \cdot n \cdot 1^n.$$

Having $\det A_1 = 2$ and $\det A_2 = 3$, we get $\det A_n = n + 1 = p$.

We compute the signature of K again through the use of the subdeterminant method where $\det A_0 = 1$:

$$\sigma(K) = \sigma(A_n) = \sum_{i=1}^{n} \operatorname{sign}(\det A_{i-1}) \cdot \operatorname{sign}(\det A_i) = \sum_{i=1}^{n} 1 = n = p - 1$$

which is the result we set out to prove. $\square$

Chapter 4

Polynomials

Historically, the first polynomial knot invariant was discovered in 1928 by J. W. Alexander. For a long time, this was the only polynomial invariant for knots. In 1984, Vaughan Jones discovered the Jones polynomial and realized that it could distinguish a knot from its mirror image, something that the Alexander polynomial could not do. This inspired more work in the area of knot invariants, the outcome of which was the strongest polynomial invariant yet, the HOMFLY polynomial.

Each of the polynomials discussed in this chapter can be constructed via something called the *skein relation*. It is a formula describing the relationship of the polynomials for three link diagrams that differ only in one crossing. In L_+ the crossing is positive, in L_- it is negative and in L_0 it is split and reconnected so that the orientation remains intact; we call it *smoothing* the crossing. Changing the sign of the crossing is called *switching*. By applying the skein relation repeatedly, we construct a binary resolving tree for a link where each parent is one of $L_\pm$ and its children are the remaining two diagrams. Eventually we have one or more trivial knots in each leaf; the process terminates then. We will see that the resolving tree exists for any link. The properties of these polynomials can be summarised as follows:

Polynomial	**Skein relation**	**Other constructions**
Alexander $\Delta_L(t)$	$\Delta_{L_+} - \Delta_{L_-} = (t^{-1/2} - t^{1/2})\Delta_{L_0}$	Seifert matrix, Alex. ideal
Conway $\nabla_L(z)$	$\nabla_{L_+} - \nabla_{L_-} = z\nabla_{L_0}$	–
Jones $V_L(t)$	$t^{-1}V_{L_+} - tV_{L_-} = (t^{1/2} - t^{-1/2})V_{L_0}$	braid representation
HOMFLY $P_L(l, m)$	$lP_{L_+} + l^{-1}P_{L_-} + mP_{L_0} = 0$	–

4.1 Alexander-Conway Polynomial

We will not show the original construction from Alexander's paper; instead, we will define the Alexander polynomial using the theory from the previous chapter.

Definition 4.1.1. (Alexander polynomial) Let M be the Seifert matrix constructed from a surface F spanning the link L. Then the Alexander polynomial for L is defined as follows

$$\Delta(t) = \det\left(t^{1/2}M - t^{-1/2}M^T\right).$$

It is a Laurent polynomial with coefficients in $\mathbb{Z}$.

The Alexander polynomial as defined above is a link invariant ([1, p.158]).

Eventually, John Conway introduced a skein relation for the Alexander polynomial (which had gone unnoticed in Alexander's original paper) and also his own version of the polynomial, the Conway polynomial. Using a simple substitution in the skein relations, one polynomial can be transformed into the other one (see below).

Definition 4.1.2. (Conway polynomial) Let L be an oriented link, denoted by $\nabla_L(z)$, we define its Conway polynomial by the three following axioms:

1. Invariance: $\nabla(L)$ is invariant under ambient isotopy of L
2. Normalisation: if L is a trivial knot then $\nabla_K(z) = 1$.
3. Skein relation: if L_+, L_- and L_0 are three oriented links with diagrams that differ only in a small neighborhood as shown in Figure 4.1, then

$$\nabla(L_+) = \nabla(L_-) + z\nabla(L_0)$$

$$\nabla(L_-) = \nabla(L_+) - z\nabla(L_0)$$

It is a polynomial in $\mathbb{Z}[z]$.

If L is a split link then $\nabla(L) = 0$ (the proof can be found in [1, p.165]).

The process of recursive application of the skein relation on a link L is called *resolution*. We can simply capture this process in a binary tree, where the parent is always one of L_+ or L_- and the children are the remaining

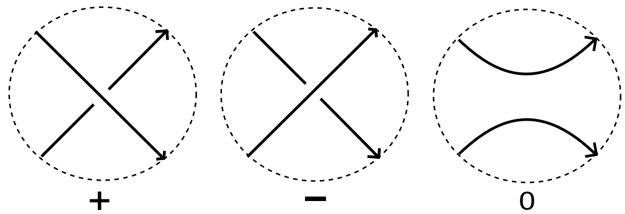

Figure 4.1: The neighborhood where the diagrams of L_+, L_- and L_0 differ.

two figures from (L_+, L_-, L_0). At the end of the resolution process, we have trivial knots or split links in the terminal nodes of the tree. The Conway polynomial of L can be expressed as a combination of the polynomials of the links in the terminal nodes.

There are many possible choices of the crossing to be changed at each point of the resolution process, and it is not obvious that the process always has to terminate.

Theorem 4.1.3. *It is possible to construct a resolving tree for a link so that in any path from the root to a terminal node, no crossing is changed more than once.*

Proof. A part of the proof is included in the solution for Exercise 4.3, the complete proof can be found in [1, p.168]. □

Theorem 4.1.4. ***(Alexander skein relation)*** *Let L_+, L_- and L_0 be three oriented links with diagrams that differ only in a small neighborhood as shown in Figure 4.1. Then*

$$\Delta(L_+) - \Delta(L_-) = (t^{-1/2} - t^{1/2})\Delta(L_0).$$

For the proof see [1, p.162].

The relationship between the two polynomials, Alexander and Conway, is following:

$$\Delta_L(x^2) = \nabla_L(x^{-1} - x).$$

It can be easily obtained by using these substitutions in their skein relations.

4.2 Jones Polynomial

Vaughan Jones discovered his polynomial while studying finite-dimensional operator algebras and n-string braid groups. We will look at a different contruction, the combinatorial approach of Luis Kauffman, and show how the Jones polynomial can be derived from his normalised bracket polynomial.

Definition 4.2.1. (bracket polynomial) Let D be an unoriented diagram of a link. We define its bracket polynomial, denoted by $\langle D \rangle$, with the following three axioms:

1. Normalisation: $\langle \bigcirc \rangle = 1$
2. Split component: $\langle D \sqcup \bigcirc \rangle = -A^{-2} - A^2 \langle D \rangle$, where $D \sqcup \bigcirc$ denotes the diagram D with a single circle that does not intersect with D
3. Skein relation:
$$\langle D_+ \rangle = A \langle D_0 \rangle + A^{-1} \langle D_\infty \rangle$$
$$\langle D_- \rangle = A \langle D_\infty \rangle + A^{-1} \langle D_0 \rangle,$$
where diagrams D_+, D_-, D_0 and D_∞ differ only in the small neighborhood shown in Figure 4.2.

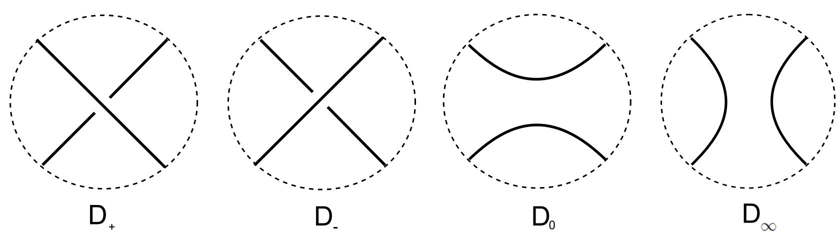

Figure 4.2: D_+, D_-, D_0 and D_∞.

The problem with this polynomial is obvious: it is defined for a particular diagram of a link, not for the link itself. The bracket polynomial is invariant under the Reidermeister moves of types II and III ([2, p.150]). However, removing a curl in a diagram changes the polynomial but not the link itself. This can be easily fixed by multiplying the polynomial with the term $(-A^{-3})$ on the power of something called the writhe of the diagram. We call the resulting polynomial the *normalised bracket polynomial*.

Definition 4.2.2. (writhe) Let D be an oriented link diagram, we assign each of its crossings $\epsilon(c) = \pm 1$, depending on its sign. We define the writhe of D as a sum over all the crossings in D:

$$w(D) = \sum_{c \in D} \epsilon(c)$$

Definition 4.2.3. (normalised bracket polynomial) The normalised bracket polynomial for an oriented diagram D is defined as follows:

$$\tilde{V}_D(A) = (-A^{-3})^{w(D)} \langle |D| \rangle (A),$$

where $|D|$ is D with its orientation ignored.

The normalised bracket polynomial is not affected by the type I Reidemeister move ([2, p.153]) and therefore does not depend on the diagram of the link by which we choose to calculate it. We denote it by $\tilde{V}_L$. We can use a simple substitution to derive the Jones polynomial from the normalised bracket polynomial.

$$V_L(t) = \tilde{V}_L(t^{-1/4}) \tag{4.1}$$

The Jones polynomial satisfies the skein relation:

$$t^{-1}V(L_+) - tV(L_-) = (t^{1/2} - t^{-1/2})V(L_0)$$

This skein relation can be derived from the skein relation for the normalised bracket polynomial using the substitution from (4.1) (see Exercise 4.5).

4.3 HOMFLY Polynomial

Seeing all the new possibilities that the discovery of the Jones polynomial brought to knot theory, more people became interested in studying polynomial invariants. Looking at the obvious similarities in the skein relations for the Jones and the Alexander polynomial, the basic idea was to find a new polynomial that would generalize both of the formerly known ones and keep all of their distinguishing abilities. The first result of this boom was a new polynomial of two variables, the HOMFLY polynomial, discovered simultaneously by two separate groups of mathematicians.

Definition 4.3.1. (HOMFLY polynomial) The HOMFLY polynomial $P_L(l, m)$ for the link L is a Laurent polynomial of two variables satisfying these conditions:

1. Invariance: $P(L)$ is a link invariant
2. Normalisation: $P(\bigcirc) = 1$
3. Skein relation: $lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0$

As stated above, the HOMFLY polynomial can be transformed into the Jones and the Alexander polynomials (see Exercise 4.7). This means that it is a link invariant stronger than both of these, but unfortunately, it is not what is called a complete invariant. There are knots that cannot be distinguished via the HOMFLY polynomial, e.g., pairs of knots called *mutant knots*. Each knot in the pair has the same HOMFLY polynomial. Details in [1, p.180] and [2, p.173].

4.4 Exercises

Exercise 4.1. ([1, p.187] Ex. 7.12.3.)

Use the Seifert matrix from Exercise 3.1 to find a formula for the Alexander polynomial of the (p, q, r) pretzel knot K with p, q and r all odd. Which pretzel knots have $\Delta(t) = 1$?

Solution. The Seifert matrix for a (p, q, r) pretzel knot is $M = \begin{pmatrix} \frac{p+q}{2} & \frac{q+1}{2} \\ \frac{q-1}{2} & \frac{q+r}{2} \end{pmatrix}$. The calculation of the Alexander polynomial goes as follows:

$$\Delta_K(t) = \det\left(t^{1/2}M - t^{-1/2}M^T\right) = \det \begin{pmatrix} \frac{(p+q)(t^{1/2}-t^{-1/2})}{2} & \frac{(q+1)t^{1/2}-(q-1)t^{-1/2}}{2} \\ \frac{(q-1)t^{1/2}-(q+1)t^{-1/2}}{2} & \frac{(q+r)(t^{1/2}-t^{-1/2})}{2} \end{pmatrix}$$

We compute this determinant from the definition and we get:

$$\Delta_K(t) = \frac{1}{4}t(pq+pr+qr+1) - \frac{1}{2}(pq+pr+qr-1) + \frac{1}{4}t^{-1}(pq+pr+qr+1)$$

For $\Delta_K(t) = 1$ we need the coefficients for t and t^{-1} to disappear and the absolute term to be one:

$$pq + pr + qr + 1 = 0$$

$$\frac{1}{2}(pq + pr + qr - 1) = 1$$

A few basic modification show that these two equations are the identical:

$$pq + pr + qr = -1$$

for which we can use the result obtained in the Exercise 3.2.

□

Exercise 4.2. ([1, p.187] Ex. 7.12.6.)

Show that if K is a knot, then the constant term of $\nabla(K)$ is 1.

Solution. Putting $z = 0$ in $\nabla_K(z)$ gives us the constant term of ∇_K. It modifies the skein relation in the following way:

$$\begin{aligned} \nabla_{K^+}(0) &= \nabla_{K^-}(0) + 0 \cdot \nabla_{K^0}(0), \text{ and so} \\ \nabla_{K^+}(0) &= \nabla_{K^-}(0) \end{aligned}$$

We know that for every knot there is a resolving tree for which no crossing is changed more than once. All the branches of the tree that include smoothing a crossing are multiplied by zero. That means that the only part of the resolving tree that is left is the path from the root to the leaf with switchings and no smoothings. Switching a crossing does not increase the number of components; no strings are cut and reattached. Because we have a 1-component knot in the root, there must be a trivial knot in the terminal node:

$$\nabla_{K_0}(0) = \nabla_{K_1}(0) = \ldots = \nabla_{K_i}(0) = \ldots = \nabla_{\bigcirc}(0) = 1,$$

where K_i is K with i crossings switched. □

Exercise 4.3. ([1, p.187] Ex. 7.12.10.)

An almost positive link is one that has a diagram in which every crossing except one is positive. Show that an almost positive link has a positive Conway polynomial (a polynomial where all the coefficients are positive).

Solution. According to Theorem 4.1.3., it is possible to construct a resolving tree for a link L so that in any path from the root to the terminal node, no crossing is changed more than once. Our goal is to show that the only negative crossing does not have to be changed at all and therefore the skein relation with a negative sign will not be used in the resolution process.

The resolving tree is constructed as follows. First we choose ordering of the components of L. For each component we choose a basepoint distinct from any crossings in the diagram and as we travel along the string in the direction of its orientation we color the crossings we pass through. If we encounter an uncolored under-crossing or a colored over-crossing, we continue. Once we get to an uncolored over-crossing, we stop – this is the crossing we want to change. Both smoothing and switching this crossing yield diagrams with fewer crossings than the original diagram without changing any of the colored crossings (this is not obvious, the complete proof can be found in [1, p.168]). We can repeat this step recursively until we get diagrams with no crossings.

If L has more than one component, we number the components so that the component with one negative crossing c_1 is L_1. We choose the basepoint on L_1 so that the strand first passes c_1 as an under-crossing, therefore c_n is colored and never changed during the resolution process. □

Exercise 4.4. ([2, p.154] Ex. 6.6)

Let L be a split link consisting of the distant union of two links L_1 and L_2. Determine how the normalised bracket polynomial $\tilde{V}(L)$ is related to $\tilde{V}(L_1)$ and $\tilde{V}(L_2)$.

Solution. First we determine the bracket polynomial for the given diagram of L. Let us denote the diagram of L, L_1 and L_2 by D, D_1 and D_2 respectively. We find the bracket polynomial for D_2, $\langle D_2 \rangle$ and put it aside. Then we take again D and we apply the skein relation repeatedly only on the crossings in D_1. Eventually we get a partial resolving tree for D which has D_2 and one or more disjoint trivial knots in all the terminal nodes. We apply the split component rule on the nodes with more than one trivial component to get only D_2 and one trivial knot in each terminal node.

Let us look at what would happen if we had only D_1. The resolution process would have been identical and we would have trivial knots in each terminal node. Now we would substitute the polynomial of the unknots by ones and rewrite the resolving tree to the polynomial form. We could do this only because we know that value of the polynomial for the knots in the terminal nodes: $\langle \bigcirc \rangle = 1$. But in fact we do know the polynomial in the terminal nodes of the partial resolving tree for D because $\langle D_2 \sqcup \bigcirc \rangle = -(A^2 + A^{-2}) \langle D_2 \rangle$ and $\langle D_2 \rangle$ was computed in the beginning. This means that instead of multiplying all the terms of the polynomial for D_1 by one, we multiply them by $-(A^2 + A^{-2}) \langle D_2 \rangle$. Now we can factor it and we get the following:

$$\langle D \rangle = -(A^2 + A^{-2}) \langle D_1 \rangle \cdot \langle D_2 \rangle$$

If this equality should stand also for the normalised bracket polynomial, we would need the writhe of D to be the sum of the writhes of its components. That is of course the case, and because the normalised bracket polynomial is invariant of the link diagram, we get:

$$\tilde{V}(L) = -(A^2 + A^{-2}) \tilde{V}(L_1) \tilde{V}(L_2)$$

□

Exercise 4.5. ([2, p.155] Ex. 6.8)

Use the skein relation of the bracket polynomial to deduce the skein relation for the Jones polynomial.

Solution. Let D_+, D_-, D_0 and D_∞ be diagrams that differ only in a small area as shown in Figure 4.2. If we multiply the skein relation for D_+ by A, the skein relation for D_- by A^{-1} and then subtract them, the skein relation for the Jones polynomial can be derived through the following set of equations:

$$\begin{aligned} A\langle D_+\rangle - A^{-1}\langle D_-\rangle &= A^2\langle D_0\rangle + \langle D_\infty\rangle - \langle D_\infty\rangle - A^{-2}\langle D_0\rangle \\ &= (A^2 - A^{-2})\langle D_0\rangle \end{aligned}$$

Now we multiply both sides by $(-A^{-3})$ on the power of the writhe of D_0:

$$A(-A^{-3})^{w(D_0)}\langle D_+\rangle - A^{-1}(-A^{-3})^{w(D_0)}\langle D_-\rangle = (A^2 - A^{-2})(-A^{-3})^{w(D_0)}\langle D_0\rangle$$

We know that $w(D_+)$ is one greater and $w(D_-)$ one less than $w(D_0)$:

$$(-A^4)\underbrace{(-A^{-3})^{w(D_0)+1}\langle D_+\rangle}_{=\tilde{V}(D_+)(A)} + A^{-4}\underbrace{(-A^{-3})^{w(D_0)-1}\langle D_-\rangle}_{=\tilde{V}(D_-)(A)} = \underbrace{(A^2 - A^{-2})(-A^{-3})^{w(D_0)}}_{=\tilde{V}(D_0)(A)}\langle D_0\rangle$$

Substituting A for $t^{-1/4}$ gives us the skein relation for the Jones polynomial:

$$t^{-1}V(D_+) - tV(D_-) = (t^{1/2} - t^{-1/2})V(D_0)$$

$\square$

Exercise 4.6. ([2, p.171] Ex. 6.20 (a))

Show that $P(L \sqcup O) = -(l + l^{-1})m^{-1}P(L)$.

Solution. Let L_+ and L_- be the original link L with an extra curl added to it, as shown in Figure 4.3. Then both L_+ and L_- are equivalent to L (because the curl can be removed by the type I Reidemeister move) which means that their HOMFLY polynomials are identical:

$$P(L_+) = P(L_-) = P(L).$$

However, smoothing this additional crossing results in separating a trivial knot from L: $L_0 = L \sqcup \bigcirc$. From the skein relation for the HOMFLY polynomial we get the following:

$$lP(L) + l^{-1}P(L) + mP(L \sqcup \bigcirc) = 0,$$

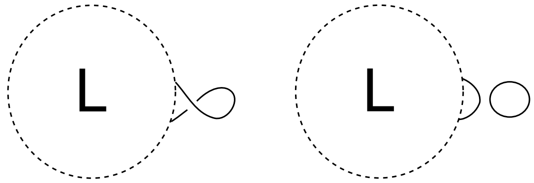

Figure 4.3: L with an extra curl and with a separated trivial knot.

after subtracting $mP(L \sqcup \bigcirc)$ and multiplying by $-m^{-1}$ we get:

$$P(L \sqcup \bigcirc) = -(l + l^{-1})m^{-1}P(L),$$

which is the desired equality. $\square$

Exercise 4.7. ([2, p.174] Ex. 6.21)
Show that the substitution

- $l = i$ and $m = i(t^{1/2} - t^{-1/2})$ turns the HOMFLY polynomial into the Alexander polynomial.
- $l = it^{-1}$ and $m = i(t^{-1/2} - t^{1/2})$ turns the HOMFLY polynomial into the Jones polynomial.

Solution. First let us look at the skein relation for the Alexander polynomial. We use the suggested substitution:

$$0 = iP(L_+) + i^{-1}P(L_-) + i(t^{1/2} - t^{-1/2})P(L_0),$$

multiplying by $-i$ gives us:

$$0 = -i^2P(L_+) + i^{-1} \cdot (-i)P(L_-) - i^2(t^{1/2} - t^{-1/2})P(L_0),$$

which is exactly the skein relation for the Alexander polynomial:

$$0 = P(L_+) - P(L_-) + (t^{1/2} - t^{-1/2})P(L_0).$$

The skein relation for the Jones polynomial is derived similarly. Again, we use the given substitution:

$$0 = it^{-1}P(L_+) + i^{-1}tP(L_-) + i(t^{-1/2} - t^{1/2})P(L_0),$$

multiply by $-i$:

$$0 = (-i^2)t^{-1}P(L_+) + (-i) \cdot i^{-1}tP(L_-) - i^2(t^{-1/2} - t^{1/2})P(L_0),$$

and get the desired skein relation:

$$0 = t^{-1}P(L_+) - tP(L_-) - (t^{1/2} - t^{-1/2})P(L_0).$$

$\square$

Bibliography

[1] Cromwell, Peter R.: *Knots and Links*, Cambridge University Press, 2004.

[2] Adams, Colin C.: *The Knot Book*, American Mathematical Society, Providence, Rhode Island, 2004.